ELSEVIER

Journal of Geometry and Physics 28 (1998) 67–84

JOURNAL OF GEOMETRY AND PHYSICS

On the spectrum of the Dirac operator under boundary conditions

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Received 4 June 1997; received in revised form 5 March 1998

Abstract

Given a Dirac operator P on a manifold with boundary, we discuss a particular local elliptic boundary condition for P as well as the (pseudo-differential) boundary condition of Atiyah–Patodi–Singer type. We prove that P is elliptic under either of these boundary conditions and extends to a self-adjoint operator with a discrete spectrum. Basic spectral estimates are given. In order to do so, we require purely functional analytic arguments and elementary estimates. © 1998 Published by Elsevier Science B.V. All rights reserved.

Subj. Class.: Quantum field theory
1991 MSC: 34L40; 81Q99
Keywords: Dirac operator; Spectrum; Boundary conditions

0. Introduction

Let $(V, \langle\ ,\ \rangle, \gamma, \nabla)$ be a Dirac bundle over a compact oriented Riemannian manifold M with boundary which need not be product near ∂M. This means that $\pi : V \to M$ is a complex vector bundle furnished with a Hermitian structure $\langle\cdot,\cdot\rangle$, a Clifford module structure $\gamma : \mathrm{Clif}(M) \to \mathrm{End}(V)$ and a compatible connection ∇. The Dirac operator P on V reads in terms of a local orthonormal frame $\{e_1, \dots, e_n\}$ for $TM|_U$ as

$$P|_U = \gamma(e_j)\nabla_{e_j}. \tag{0.1}$$

The aim of this paper is to establish elliptic boundary conditions for the Dirac operator P and study its spectrum. This can be done easily, if there exists a chirality operator F on V, i.e.

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PII: S0393-0440(98)00013-8

a self-adjoint isomorphism $F \in \mathrm{End}(V)$ which is covariantly constant and anti-commutes with the Clifford map γ. In that case the local boundary conditions, given by

$$\phi|_{\partial M} \in \Gamma_{\pm}^{\mathrm{loc}}, \quad \text{meaning that} \quad (F\gamma(N)\phi)|_{\partial M} = \pm\phi|_{\partial M}, \tag{0.2}$$

turn out to be elliptic. Here N denotes the unit normal on ∂M.

On the other hand P can be decomposed on the boundary ∂M into a normal and tangential component, reading

$$(P\psi)|_U = \gamma(N)(\nabla_N\psi + A\psi)|_U. \tag{0.3}$$

The operator A is L^2 self-adjoint on ∂M and has a discrete spectrum, which (eventually after a small perturbation) is bounded away from 0. The famous pseudo-differential boundary condition of Atiyah–Patodi–Singer type then requires that the restriction $\phi|_{\partial M}$ is in the negative eigenspace of the operator A, i.e., $\phi|_{\partial M} \in \Gamma_{-}^{\mathrm{APS}}$. It is well known that this non-local boundary condition is elliptic.

Technically, our approach is based on the Lichnerowicz formula, which states that P^2 is the connection Laplacian $\nabla^*\nabla$ perturbed by a curvature endomorphism, say $\mathcal{R} \in \mathrm{End}(V)$, i.e.

$$P^2\psi = \nabla^*(\nabla\psi) + \mathcal{R}\psi. \tag{0.4}$$

Based on this, the key observation is to prove that P satisfies under either of the boundary conditions above the elliptic estimate

$$\|\psi\|_{H^1}^2 \le C_0\|P\psi\|_{L^2}^2 \quad \forall\psi \in (\mathcal{N}_0^{\perp}(P) \cap \mathcal{D}_{-}(P)). \tag{0.5}$$

Here $\mathcal{N}_0^{\perp}(P)$ denotes the complement of the kernel of the Dirac operator P. Then our main results read as follows:

- The Dirac operator P with domain $\mathcal{D}_{-}(P)$ is a self-adjoint operator on $L^2(M, V)$. Here $\mathcal{D}_{-}(P)$ is the space of all sections in $H^1(M, V)$ which satisfy the boundary condition $\phi|_{\partial M} \in \Gamma_{-}^{\mathrm{loc}}$ or $\phi|_{\partial M} \in \Gamma_{-}^{\mathrm{APS}}$, respectively.
- The operator P with domain $\mathcal{D}_{-}(P)$ has a pure point spectrum. The eigensections $\phi \in \mathcal{D}_{-}(P)$, satisfying $P\phi = \lambda\phi$ are smooth.
- If the mean curvature of the boundary $\partial M \hookrightarrow M$ is strictly positive, then the curvature endomorphism $\mathcal{R}$ gives a lower bound for λ_1, the eigenvalue of P on $\mathcal{D}_{-}(P)$ of least absolute value. That is

$$(\lambda_1)^2 \ge \min_{p\in M} \rho_0(p) \quad \text{where} \quad \rho_0(p) = \inf_{v\in V_p} \frac{\langle v, \mathcal{R}v\rangle}{\langle v, v\rangle}. \tag{0.6}$$

Most of these results are well known. In particular the (non-local) Atiyah–Patodi–Singer boundary condition has been discussed a lot in the literature, cf. [4,5] for general results, and [12] for original reference. In the present paper we show that the results on the Dirac operator on manifolds with boundary can be established as well on a very elementary level. We present an approach to tackle this type of problems completely by means of basic estimate and standard results from functional analysis. In particular no reference is needed to the calculus of (elliptic) pseudo-differential boundary problems, cf. [6] or [3].

In Sections 1 and 2 we give a short review on Dirac bundles and establish a global Lichnerowicz formula. This is the key for proving a priori estimates for P under either of the boundary conditions $\phi|_{\partial M} \in \Gamma^{\text{loc}}_{\pm}$ and $\phi|_{\partial M} \in \Gamma^{\text{APS}}_{\pm}$. This is worked out in Sections 3 and 4. In Section 5 we show that P is self-adjoint and allows for the estimate (0.5). In Section 6 we show that P is a Fredholm operator which has discrete spectral resolution. Moreover we establish the (geometric) estimate (0.6) for the first eigenvalue. Regularity results are given in Section 7. Section 8 contains an application to inhomogeneous boundary value problems and a Hodge-type decomposition.

1. Dirac bundles

Let (M, g) be an orientable n-dimensional C^∞ Riemannian manifold with smooth boundary ∂M. The tangent and cotangent bundle are identified by the $\flat$-map, $v^\flat(w) := g(v, w)$. Its inverse is denoted by $\sharp$. The Clifford bundle $\text{Clif}(M)$ is defined by the relation

$$v * w + w * v = -2g(v, w)\text{Id}. \tag{1.1}$$

Let $\pi : V \to M$ be a smooth complex vector bundle furnished with a Hermitian structure $\langle \cdot, \cdot \rangle$. The space $C^\infty(M, V)$ of all smooth sections is equipped with a L^2-structure

$$\langle\langle \phi, \psi \rangle\rangle := \int_M \langle \phi, \psi \rangle \, d\mu, \tag{1.2}$$

where $d\mu$ is the Riemannian volume element. $L^2(M, V)$ is the completion of $C^\infty(M, V)$ with respect to the corresponding norm $\|\phi\|_L^2$. For the restriction of the bundle V to the boundary ∂M we write V_∂. On the space of smooth sections $C^\infty(\partial M, V_\partial)$ an L^2-structure is defined accordingly by integration with respect to the induced Riemannian volume $d\mu_\partial$.

A Clifford module structure γ on V is a $\mathbb{R}$-algebra bundle morphism

$$\gamma : \text{Clif}(M) \to \text{End}(V).$$

Then the quadruple $(V, \langle , \rangle, \gamma, \nabla)$ defines a *Dirac bundle* if:

- The Clifford multiplication is fibrewise skew-adjoint, i.e.,

$$\langle \gamma(w)\phi, \psi \rangle + \langle \phi, \gamma(w)\psi \rangle = 0 \quad \forall w \in TM, \ \ \forall \phi, \psi \in C^\infty(M, V). \tag{1.3}$$

- There exists a Hermitian connection $\nabla : C^\infty(M, V) \to C^\infty(M, T^*M \otimes V)$, which acts as a module derivation with respect to the Clifford map γ, that is, $\nabla\gamma = 0$. In other words the compatible connection ∇ satisfies

$$\nabla(\gamma(w)\phi) = \gamma(\nabla w)\phi + \gamma(w)\nabla\phi$$
$$\forall w \in C^\infty(M, \text{Clif}(M)), \ \ \forall \phi \in C^\infty(M, V), \tag{1.4}$$

where ∇w is understood as the action of the induced Levi-Civitá connection on $\text{Clif}(M)$.

For each Clifford module $\gamma : \mathrm{Clif}(M) \to \mathrm{End}(V)$ one can construct a corresponding Dirac bundle, cf. [2]. That is, one can always find:

- a Hermitian structure $\langle\ ,\ \rangle$ which makes the Clifford multiplication by tangent vectors $\gamma(v)$ skew-adjoint,
- a locally and globally well-defined connection ∇, which is Hermitian and compatible.

In general there are several possible choices for the connection ∇ which makes $(V, \langle,\rangle, \gamma)$ into a Dirac bundle. However, there exists a unique *spin connection* ∇^S which is compatible and an extension of the Levi-Civitá connection in the following sense.

It suffices to define ∇^S locally and patch it together by a partition of unity, since a convex combination of compatible connections is compatible. To do so, we utilize the connection matrix ω^S, which a 1-form on $U \subset M$ with values in $\mathrm{End}(V|_U)$. Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame for $TM|_U$ and $\{s_1, \dots, s_N\}$, a local bundle frame on $V|_U$, then each $s \in C^\infty(M, V)$ locally writes as $s = f^J s_J$. If Γ^i_{jk} denote the Christoffel symbols for the Levi-Civitá connection on U, the connection 1-form ω^S is defined by

$$\omega^S(e_k) := \tfrac{1}{4}\Gamma^i_{jk}\gamma(e_j)\gamma(e_i). \tag{1.5}$$

Then the spin connection ∇^S is (locally) defined by

$$\nabla^S s = df^J \otimes s_J + f^J \nabla^S s_J, \qquad \text{where } \nabla^S s_J = (\omega^S)^K_J \otimes s_K. \tag{1.6}$$

This spin connection satisfies conditions (1.3) and (1.4) needed to define a Dirac bundle.

If $(V, \langle,\rangle, \gamma, \nabla)$ be a Dirac bundle, a smooth self-adjoint isomorphism $F \in \mathrm{End}(V)$ is called a *chirality operator*, if

$$F \circ \gamma(v) + \gamma(v) \circ F = 0 \qquad \nabla_v F = 0, \quad \forall\, v \in TM, \text{ and } F^2 = \mathrm{Id}. \tag{1.7}$$

In general, such a structure need not exist, but there are topological obstructions, cf. [4]. However, if M is even-dimensional, the normalized orientation can be used to define such an isomorphism. Therefore let $\{e_1, \dots, e_n\}$ is a local orthonormal on $U \subset M$, then the operator

$$F|_U := (\sqrt{-1})^{n/2}\gamma(e_1)\gamma(e_2)\cdots\gamma(e_n) \tag{1.8}$$

is well-defined locally, i.e. independent of the choice of the frame. By partition of unity we can construct a globally well-defined chirality operator F.

If $(V, \langle,\rangle, \gamma, \nabla)$ is a Dirac bundle over the Riemannian manifold (M, g), the *Dirac operator* P is defined by the composition

$$C^\infty(M, V) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes V) \xrightarrow{\sharp \otimes \mathrm{Id}} C^\infty(M, TM \otimes V) \xrightarrow{\gamma} C^\infty(M, V).$$

That is

$$\begin{aligned} P : C^\infty(M, V) &\longrightarrow C^\infty(M, V) \\ \phi &\mapsto P(\phi) = \gamma \circ (\sharp \otimes \mathrm{Id})(\nabla\phi). \end{aligned} \tag{1.9}$$

The leading symbol is given by the Clifford multiplication as $\sigma_L(P)(x,\xi) = \sqrt{-1}\,\gamma(\xi^\sharp)$. In terms of a local orthonormal frame $\{e_1, \dots, e_n\}$ the action of P becomes

$$(P\phi)|_U = \gamma(e_j)(\nabla_{e_j}\phi) \qquad \forall \phi \in C^\infty(M, V). \tag{1.10}$$

If F is a chirality operator on $(V, \langle , \rangle, \gamma, \nabla)$, it anti-commutes with the Dirac operator, i.e.

$$PF + FP = 0. \tag{1.11}$$

2. A global Lichnerowicz formula

The square of the Dirac operator $P^2 : C^\infty(M, V) \to C^\infty(M, V)$ defines the *Dirac–Laplace operator*. This second order operator can be rewritten in terms of the connection Laplacian on V by means of a Lichnerowicz formula reading

$$P^2\psi = \nabla^*\nabla\psi + \mathcal{R}\psi, \tag{2.1}$$

where $\mathcal{R}$ stands for the curvature endomorphism, cf. [1].

In order to derive an integrated version of (2.1) we need to study the Dirac operator near the boundary. Therefore let N be the inward pointing unit normal field on ∂M, and $\{N, \bar{e}_2, \dots, \bar{e}_n\}$, a local orthonormal frame on a neighborhood $U \cap \partial M$. Then

$$(P\psi)|_U = \gamma(N)(\nabla_N\psi + A\psi)|_U \quad \text{where } A = -\gamma(N)\gamma(\bar{e}_j)\nabla_{\bar{e}_j}. \tag{2.2}$$

From (1.11) we get $AF|_{\partial M} - F|_{\partial M}A = 0$. Moreover it is straightforward to verify that

$$A\gamma(N) + \gamma(N)A = -\gamma(N)\gamma(\bar{e}_j)\gamma(\nabla_{\bar{e}_j}N) = -\gamma(N)\mathcal{S}_\partial, \tag{2.3}$$

where $\mathcal{S}_\partial$ is the mean curvature, i.e., the trace of the second fundamental form of the submanifold $\partial M \subset M$.

Let a differential 1-form $\mathcal{C}_{\phi\psi} \in C^\infty(M, T^*M)$ be given by $\mathcal{C}_{\psi\phi}(v) := \langle\gamma(v)\psi, \phi\rangle$. Its divergence d^* can be written in terms of P as

$$d^*\mathcal{C}_{\psi\phi} = -(\nabla_{e_j}\mathcal{C}_{\psi\phi}(e_j)) + \mathcal{C}_{\psi\phi}(\nabla_{e_j}e_j) = -\langle P\psi, \phi\rangle + \langle\psi, P\phi\rangle. \tag{2.4}$$

From the Gauß theorem we then infer that the Dirac operator is symmetric with respect to the L^2-structure (1.2) modulo a boundary term, i.e.

$$\langle\langle P\psi, \phi\rangle\rangle - \langle\langle\psi, P\phi\rangle\rangle = -\int_{\partial M} \langle\gamma(N)\psi, \phi\rangle \, \mathrm{d}\mu_\partial. \tag{2.5}$$

Given a local orthonormal frame $\{e_1, \dots, e_n\}$ on each $U_\alpha \in \mathcal{U}$ for an appropriate cover $\mathcal{U}$ of M, and a subordinated partition of unity ρ_α, then the H^1 inner product on $C^\infty(M, V)$ reads

$$\langle\langle\psi, \phi\rangle\rangle_{H^1} = \langle\langle\psi, \phi\rangle\rangle + \langle\langle\nabla\psi, \nabla\phi\rangle\rangle \quad \forall\, \psi, \phi \in C^\infty(M, V)$$

where

$$\langle\langle \nabla\psi, \nabla\phi \rangle\rangle = \sum_{\alpha} \int_{U_\alpha} \sum_{j=1\ldots n} \rho_\alpha \langle \nabla_{e_j}\psi, \nabla_{e_j}\phi \rangle \, \mathrm{d}\mu. \tag{2.6}$$

We can write things this way, since the expression on the right-hand side is independent of the partition and the local frame chosen.

Proposition 1. *The H^1 inner product can be computed in terms of the Dirac operator P, the curvature endomorphism $\mathcal{R}$ and the boundary operator A as*

$$\begin{aligned} \langle\langle \psi, \phi \rangle\rangle_{H^1} &= \langle\langle \psi, \phi \rangle\rangle + \langle\langle P\psi, P\phi \rangle\rangle \\ &\quad - \langle\langle \psi, \mathcal{R}\phi \rangle\rangle + \int_{\partial M} \langle \psi, A\phi \rangle \, \mathrm{d}\mu_\partial. \end{aligned} \tag{2.7}$$

Proof. Using (2.5) and the Lichnerowicz formula (2.1) we get

$$\langle\langle P\psi, P\phi \rangle\rangle = \langle\langle \psi, \nabla^*\nabla\phi \rangle\rangle + \langle\langle \psi, \mathcal{R}\phi \rangle\rangle - \int_{\partial M} \langle \gamma(N)\psi, P\phi \rangle \, \mathrm{d}\mu_\partial. \tag{2.8}$$

On the other hand the Gauß theorem implies that

$$\langle\langle \psi, \nabla^*\nabla\phi \rangle\rangle = \langle\langle \nabla\psi, \nabla\phi \rangle\rangle + \int_{\partial M} \langle \psi, \nabla_N\phi \rangle \, \mathrm{d}\mu_\partial. \tag{2.9}$$

From the splitting (2.2) of the Dirac operator on ∂M we infer that

$$\int_{\partial M} (\langle \psi, \nabla_N\phi \rangle - \langle \gamma(N)\psi, P\phi \rangle) \, \mathrm{d}\mu_\partial = - \int_{\partial M} \langle \psi, A\phi \rangle \, \mathrm{d}\mu_\partial. \tag{2.10}$$

Adding Eqs. (2.8) and (2.9) then proves the assumption. □

An obvious consequence of the preceding result is the corresponding formula which allows one to compute the H^1 norm on $C^\infty(M, V)$ in terms of P:

$$\|\nabla\psi\|_{L^2}^2 = \|P\psi\|_{L^2}^2 - \langle\langle \psi, \mathcal{R}\psi \rangle\rangle + \int_{\partial M} \langle \psi, A\psi \rangle \, \mathrm{d}\mu_\partial. \tag{2.11}$$

3. A local elliptic boundary condition for the Dirac operator

If $(V, \langle , \rangle, \gamma, \nabla)$ be a Dirac bundle with a chirality operator F it is possible to construct a local boundary condition for the Dirac operator P in terms of the operator,

$$\Gamma : V_\partial \longrightarrow V_\partial \quad \text{defined by} \Gamma := F|_{\partial M}\gamma(N). \tag{3.1}$$

From (1.1), (1.3) and (1.7) it is clear that

$$\Gamma^2 = \mathrm{Id}, \quad \Gamma F + F\Gamma = 0 \quad \langle \Gamma\phi, \psi\rangle = \langle \phi, \Gamma\psi\rangle \;\; \forall \phi, \psi \in C^\infty(\partial M, V_\partial). \tag{3.2}$$

Hence Γ has the eigenvalues ± 1, and the eigenspaces

$$\begin{aligned} \Gamma_+^{\mathrm{loc}} &= \{\phi \in C^\infty(\partial M, V_\partial) \mid \Gamma\phi = \phi\}, \\ \Gamma_-^{\mathrm{loc}} &= \{\phi \in C^\infty(\partial M, V_\partial) \mid \Gamma\phi = -\phi\} \end{aligned} \tag{3.3}$$

are orthogonal with respect to the induced Hermitian structure $\langle\, ,\, \rangle$ on V_∂. The corresponding projections $\pi_\pm := \frac{1}{2}(\mathrm{Id} \pm \Gamma)$ are also self-adjoint operators so that

$$\langle \pi_+(\phi), \pi_-(\psi)\rangle = 0 \quad \forall \phi, \psi \in C^\infty(\partial M, V_\partial). \tag{3.4}$$

From (1.7) we infer that $\Gamma\gamma(N)\phi = \mp\gamma(N)\phi$ for $\phi \in \Gamma_\pm^{\mathrm{loc}}$, respectively. Hence $\gamma(N)$ acts as an isomorphism intertwining Γ_+^{loc} and Γ_-^{loc}, i.e.,

$$\gamma(N) \circ \pi_+ = \pi_- \circ \gamma(N) \quad \text{and} \quad \gamma(N) \circ \pi_- = \pi_+ \circ \gamma(N). \tag{3.5}$$

Moreover we infer from (2.3) and (1.7) that $\Gamma A + A\Gamma = -\Gamma \mathcal{S}_\partial$, and hence

$$A \circ \pi_- = \pi_+ \circ A + \tfrac{1}{2}\mathcal{S}_\partial \Gamma. \tag{3.6}$$

Lemma 2. *Under the local boundary condition $\pi_+(\psi) = 0$ the Dirac operator P allows for an elliptic estimate. That is, for each $\delta > 0$ there exists C_δ such that ψ satisfies*

$$\|\psi\|_{H^1}^2 \le (1+\delta)\|P\psi\|_{L^2}^2 + C_\delta\|\psi\|_{L^2}^2 \quad \forall \psi \in C^\infty(M, V) \cap \Gamma_-^{\mathrm{loc}}. \tag{3.7}$$

Proof. In order to prove the estimate (3.7) we have to control the boundary integral of Eq. (2.11) under the boundary condition $\psi|_{\partial M} = \pi_-(\psi)$. Since $\pi_\pm$ are self-adjoint projections, we get by using the commutation relation (3.6)

$$\int_{\partial M} \langle \psi, A\psi\rangle \,\mathrm{d}\mu_\partial = \int_{\partial M} \langle \pi_-(\psi), A\pi_-(\psi)\rangle \,\mathrm{d}\mu_\partial = \frac{1}{2}\int_{\partial M} \mathcal{S}_\partial \langle \psi, \Gamma\psi\rangle \,\mathrm{d}\mu_\partial. \tag{3.8}$$

Therefore (2.11) turns into

$$\begin{aligned} \|P\psi\|_{L^2}^2 = {} & \|\nabla\psi\|_{L^2}^2 + \langle\langle \psi, \mathcal{R}\psi\rangle\rangle \\ & + \frac{1}{2}\int_{\partial M} \mathcal{S}_\partial \langle \psi, \psi\rangle \,\mathrm{d}\mu_\partial \quad \forall \psi \in C^\infty(M, V) \cap \Gamma_-^{\mathrm{loc}}. \end{aligned} \tag{3.9}$$

Since M is compact, $\mathcal{R} \in \mathrm{End}(V)$ and $\mathcal{S}_\partial \in C^\infty(\partial M, V_\partial)$ are bounded on M and ∂M, respectively. Moreover, the restriction to the boundary extends to a compact operator from $H^1(M, V)$ to $L^2(\partial M, V_\partial)$. Therefore an inequality of Ehrling type applies, cf. [10]. That is, for each $\epsilon > 0$ there is $C_\epsilon > 0$ such that

$$\|\psi\|_{L^2(\partial M)}^2 \le \epsilon\|\psi\|_{H^1}^2 + C_\epsilon\|\psi\|_{L^2}^2 \quad \forall \psi \in C^\infty(M, V) \tag{3.10}$$

Using this, the estimate (3.7) follows from (3.9). □

Obviously this result holds correspondingly for the Dirac operator the complementary boundary condition $\pi_-(\psi) = 0$, i.e.,

$$\|\psi\|_{H^1}^2 \leq (1+\delta)\|P\psi\|_{L^2}^2 + C_\delta\|\psi\|_{L^2}^2 \quad \forall\psi \in C^\infty(M, V) \cap \Gamma_+^{\text{loc}}. \tag{3.11}$$

Corresponding results on the Dirac operator under local boundary conditions can be found in [12], cf. also [4]. That approach, presented there, is based on the concept of strongly elliptic boundary value problems, cf. also [5]. In contrast, the line of arguments in this paper requires not more than the estimate (3.7) and some standard arguments from functional analysis.

A local boundary conditions of the type (3.3) which implies the estimate (3.7) can also be established in cases where a chirality operator F does not exist. In [11] we studied a Lorentz manifold $N = \mathbb{R} \times M$, where M is a compact Riemannian manifold with boundary. With e_0 denoting a time-like unit vector the operator $\widetilde{\Gamma} := \gamma(e_0)\gamma(N)$ can be used to construct an elliptic boundary condition for the induced Dirac operator $\widetilde{P}$ on M. However, that case does not fit completely into the discussion of this paper, since $\widetilde{P}$ does not correspond to a Dirac bundle $(V, \langle\ ,\ \rangle, \gamma, \nabla)$ on M.

4. A boundary condition of Atiyah–Patodi–Singer type

In the context of index theorems another non-local boundary condition for the Dirac operator P is more common. In this section we consider a modified version of this Atiyah–Patodi–Singer boundary condition. By (2.2), $P|_{\partial M} = \gamma(N)(\nabla_N + A)$ where A is self-adjoint with respect to the L^2 structure on the boundary. For $\epsilon > 0$ let

$$A_\epsilon := A + \tfrac{1}{2}\mathcal{S}_\partial \,\mathrm{Id} + \epsilon F : C^\infty(\partial M, V_\partial) \longrightarrow C^\infty(\partial M, V_\partial) \tag{4.1}$$

be the perturbation of A, where $\mathcal{S}_\partial$ is the mean curvature and F is the chirality operator. Then A_ϵ is a self-adjoint, too. From (1.7) and (2.3) we read of the commutation relations

$$A_\epsilon\gamma(N) + \gamma(N)A_\epsilon = 0 \quad \text{and} \quad A_\epsilon F - FA_\epsilon = 0. \tag{4.2}$$

Therefore A_ϵ has a discrete symmetric spectrum, and for arbitrary small $\epsilon > 0$ we can guarantee that $\ker(A_\epsilon) = 0$. Let $(\varphi_k)_{k\in\mathbb{N}}$ be the spectral resolution of A_ϵ, i.e. $A_\epsilon\varphi_k = \lambda_k\varphi_k$. The corresponding positive and negative eigenspaces of A_ϵ we denote by

$$\begin{aligned} \Gamma_+^{\text{APS}} &= \left\{\phi \in C^\infty(\partial M, V_\partial) \mid \phi = \sum_{\lambda_k>0} c_k\varphi_k\right\}, \\ \Gamma_-^{\text{APS}} &= \left\{\phi \in C^\infty(\partial M, V_\partial) \mid \phi = \sum_{\lambda_k<0} c_k\varphi_k\right\}. \end{aligned} \tag{4.3}$$

For the projections onto these spaces we write π_+ and π_-.

Lemma 3. *If $(V, \langle,\rangle, \gamma, \nabla)$ is a Dirac bundle with chirality F, then $\pi_+(\psi) = 0$ is an elliptic boundary condition for P. For each $\delta > 0$ there exists C_δ such that ψ satisfies a Friedrichs estimate*

$$\|\psi\|^2_{H^1} \le (1+\delta)\|P\psi\|^2_{L^2} + C_\delta\|\psi\|^2_{L^2} \quad \forall\psi \in C^\infty(M,V) \cap \Gamma^{\mathrm{APS}}_{-}. \tag{4.4}$$

Proof. From (2.11) we infer that

$$\|\nabla\psi\|^2_{L^2} = \|P\psi\|^2_{L^2} - \langle\langle\psi, \mathcal{R}\psi\rangle\rangle + \int_{\partial M} \langle\psi, A_\epsilon\psi\rangle \,\mathrm{d}\mu_\partial - \frac{1}{2}\int_{\partial M} \mathcal{S}_\partial\langle\psi,\psi\rangle \,\mathrm{d}\mu_\partial - \epsilon\int_{\partial M}\langle\psi, F\psi\rangle\,\mathrm{d}\mu_\partial \tag{4.5}$$

for all $\psi \in C^\infty(M,V)$. Under the boundary condition $\pi_+(\psi) = 0$ we have

$$\int_{\partial M} \langle\psi, A_\epsilon\psi\rangle\,\mathrm{d}\mu_\partial = \sum_{\lambda_n<0} |c_n|^2\lambda_n \le 0 \quad \forall\psi \in \Gamma^{\mathrm{APS}}_{-}. \tag{4.6}$$

Therefore

$$\|\psi\|^2_{H^1} \le C\|\psi\|^2_{L^2} + \|P\psi\|^2_{L^2} - \int_{\partial M}\langle\psi, (\tfrac{1}{2}\mathcal{S}_\partial + \epsilon F)\psi\rangle\,\mathrm{d}\mu_\partial. \tag{4.7}$$

Estimating this boundary integral as in proof of (3.7) then proves the assumption. □

It is well known that elliptic boundary conditions of the Atiyah–Patodi–Singer type does not at all depend on the existence of a chirality operator F. For this paper, however, we specialize to the particular condition (4.3) in order to treat the case $\psi \in \Gamma^{\mathrm{APS}}_{-}$ simultaneously with the local boundary condition $\psi \in \Gamma^{\mathrm{loc}}_{-}$.

5. Self-adjointness and elliptic estimates

On the basis of the estimate of the preceding sections we are able to treat boundary value problems for the Dirac operator under the respective boundary conditions $\psi \in \Gamma^{\mathrm{loc}}_{-}$ and $\psi \in \Gamma^{\mathrm{APS}}_{-}$. We observe that the Dirac operator extends to a bounded linear operator $P : H^1(M,V) \to L^2(M,V)$, where $H^1(M,V)$ is completion of $C^\infty(M,V)$ in the norm (2.6). The restriction to the boundary is a compact linear map from $H^1(M,V)$ to $L^2(\partial M, V_\partial)$, the completion of $C^\infty(\partial M, V_\partial)$. By construction we have an orthonormal decomposition on the boundary, reading

$$L^2(\partial M, V_\partial) = L^2\Gamma_+ \oplus L^2\Gamma_-. \tag{5.1}$$

Here – and in the sequel – we use the symbol $\Gamma_\pm$ for $\Gamma^{\mathrm{loc}}_{\pm}$ and $\Gamma^{\mathrm{APS}}_{\pm}$, simultaneously.

Theorem 4. *Let $(V, \langle,\rangle, \gamma, \nabla)$ be a Dirac bundle with chirality F. Then the Dirac operator P extends to a self-adjoint linear operator on $L^2(M,V)$ with domain*

$$\mathcal{D}_-(P) := \{\phi \in H^1(M,V) \mid \phi|_{\partial M} \in L^2\Gamma_-\}. \tag{5.2}$$

Proof. From (2.5) we infer that

$$\langle\langle P\phi, \psi\rangle\rangle - \langle\langle \phi, P\psi\rangle\rangle = -\int_{\partial M} \langle \gamma(N)\phi, \psi\rangle \, \mathrm{d}\mu_\partial \quad \forall \psi, \phi \in H^1(M, V). \tag{5.3}$$

In particular, if $\phi, \psi \in \mathcal{D}_-(P)$ then $\gamma(N)\phi \in L^2\Gamma_+$, cf. (3.5) and (4.2). The orthogonality of the splitting (5.1) implies that P is a symmetric operator.

Let P^* be the adjoint operator. Its domain is

$$\mathcal{D}_-(P^*) = \{\theta \in L^2(M, V) \mid \exists \chi \in L^2(M, V) \text{ with } \langle\langle \chi, \psi\rangle\rangle = \langle\langle \theta, P\psi\rangle\rangle \quad \forall \psi \in \mathcal{D}_-(P)\}. \tag{5.4}$$

Hence for each $\overline{\theta} \in \mathcal{D}_-(P^*) \cap H^1(M, V)$ there exists $\chi \in L^2(M, V)$ such that

$$\langle\langle \chi, \psi\rangle\rangle = \langle\langle P\overline{\theta}, \psi\rangle\rangle + \int_{\partial M} \langle \gamma(N)\overline{\theta}, \psi\rangle \, \mathrm{d}\mu_\partial \quad \forall \psi \in \mathcal{D}_-(P). \tag{5.5}$$

In particular we may choose ψ to be supported away from ∂M, i.e. $\psi \in C_0^\infty(M, V)$. Since $C_0^\infty(M, V) \subset L^2(M, V)$ is dense, this implies that

$$\chi = P\overline{\theta} \quad \text{and} \quad \int_{\partial M} \langle \gamma(N)\overline{\theta}, \psi\rangle \, \mathrm{d}\mu_\partial = 0 \ \forall \psi \in \mathcal{D}_-(P). \tag{5.6}$$

From (3.5) and (4.2) we infer that $\overline{\theta}|_{\partial M} \in L^2\Gamma_-$. This proves that

$$\mathcal{D}_-(P^*) \cap H^1(M, V) = \mathcal{D}_-(P). \tag{5.7}$$

For a general $\theta \in \mathcal{D}_-(P^*)$ there exists a sequence $\theta_j \in (\mathcal{D}_-(P^*) \cap H^1(M, V))$ such that $\theta_j \to \theta$ (strongly) in $L^2(M, V)$. Then

$$\langle\langle P\theta_j, \psi\rangle\rangle = \langle\langle \theta_j, P\psi\rangle\rangle \longrightarrow \langle\langle \theta, P\psi\rangle\rangle = \langle\langle P^*\theta, \psi\rangle\rangle \quad \forall \psi \in \mathcal{D}_-(P), \tag{5.8}$$

which implies that $P\theta_j$ is weakly convergent in $L^2(M, V)$. In particular the sequence is bounded, and the estimates (3.7) and (4.4) yield

$$\|\theta_j\|^2_{H^1} \leq (1 + \delta)\|P\theta_j\|^2_{L^2} + C_\delta \|\theta_j\|^2_{L^2} \leq K. \tag{5.9}$$

Hence there exists a weakly convergent subsequence $\theta_{j_k} \rightharpoonup \widehat{\theta}$ in $H^1(M, V)$. The uniqueness of the weak limit implies that $\widehat{\theta} = \theta$. This proves that $\theta \in \mathcal{D}_-(P^*) \cap H^1(M, V)$, and hence P is a self-adjoint. □

Knowing about the self-adjointness of the Dirac operator with domain $\mathcal{D}_-(P)$ it is clear that its spectrum is real. We denote by the eigenspaces

$$\mathcal{N}_\lambda(P) := \{\psi \in \mathcal{D}_-(P) \mid (P - \lambda)\psi = 0\}. \tag{5.10}$$

Of particular interest is the kernel of P, i.e. the eigenspace $\mathcal{N}_0(P)$.

Lemma 5. *The space $\mathcal{N}_0(P)$ is finite dimensional.*

Proof. The estimates (3.7) and (4.4) imply that

$$\|\psi\|^2_{H^1} \leq C_\delta \|\psi\|^2_{L^2} \quad \forall \psi \in \mathcal{N}_0(P). \tag{5.11}$$

Let ψ_j be an arbitrary sequence in the unit disk $D^1_{\mathcal{N}_0} := \{\psi \in \mathcal{N}_0(P) \mid \|\psi\|^2_{H^1} \leq 1\}$. By Rellich's lemma there is a convergent subsequence $\psi_{j_k} \to \widehat{\psi}$ in $L^2(M, V)$, which is a H^1-Cauchy sequence by (5.11). Therefore $D^1_{\mathcal{N}_0}$ is compact in $H^1(M, V)$, and hence $\mathcal{N}_0(P)$ is finite-dimensional. □

In particular we infer from this lemma that $\mathcal{N}_0(P)$ is a closed subspace of $\mathcal{D}_-(P)$, so that we have an orthogonal decomposition

$$\mathcal{D}_-(P) = \mathcal{N}_0(P) \oplus_{L^2} (\mathcal{N}_0^\perp(P) \cap \mathcal{D}_-(P)). \tag{5.12}$$

Lemma 6. *There exists a universal constant $C_0 \in [1, \infty]$ such that*

$$\|\psi\|^2_{H^1} \leq C_0 \|P\psi\|^2_{L^2} \quad \forall \psi \in (\mathcal{N}_0^\perp(P) \cap \mathcal{D}_-(P)). \tag{5.13}$$

Proof. Let ϕ_j be a minimising sequence for the quadratic form $\|P\psi\|^2_{L^2}$ in the unit sphere

$$S^1_{\mathcal{N}_0^\perp} := \{\widehat{\psi} \in (\mathcal{N}_0^\perp(P) \cap \mathcal{D}_-(P)) \mid \|\widehat{\psi}\|^2_{L^2} = 1\}\,. \tag{5.14}$$

By (3.7) and (4.4) this sequence is bounded in $H^1(M, V)$ and hence has a weakly convergent subsequence $\phi_{j_k} \rightharpoonup \widehat{\phi}$. The functional $\|P\psi\|^2_{L^2}$ is weakly lower semicontinuous on $H^1(M, V)$ and it follows that $\|P\widehat{\phi}\|^2_{L^2} \leq \|P\widehat{\psi}\|^2_{L^2}$ for all $\widehat{\psi} \in S^1_{\mathcal{N}_0^\perp}$. Therefore

$$\|\psi\|^2_{L^2} \|P\widehat{\phi}\|^2_{L^2} \leq \|P\psi\|^2_{L^2} \quad \forall \psi \in (\mathcal{N}^\perp(P) \cap \mathcal{D}_-(P)). \tag{5.15}$$

On the other hand the embedding $H^1(M, V) \hookrightarrow L^2(M, V)$ is compact, so that $\phi_j \to \phi_\infty$ (strongly) in $L^2(M, V)$, up to the selection of a subsequence. Thus $\|\phi_\infty\|_{L^2} = 1$, and from the uniqueness of the weak L^2-limit we infer that $\phi_\infty = \widehat{\phi} \in S^1_{\mathcal{N}_0^\perp}$. Therefore $\|P\widehat{\phi}\|^2_{L^2} \neq 0$ and the estimates (3.8) and (4.4) turn into

$$\|\psi\|^2_{H^1} \leq \|P\psi\|^2_{L^2} \left(1 + \delta + \frac{C_\delta}{\|P\widehat{\phi}\|^2_{L^2}}\right) = C_0 \|P\psi\|^2_{L^2}$$
$$\forall \psi \in (\mathcal{N}^\perp(P) \cap \mathcal{D}_-(P)), \tag{5.16}$$

which proves the assertion. □

6. The spectrum of Dirac operator

Having access to the estimate (5.13) is the key to study the analysis of the Dirac operator under the boundary conditions imposed. In particular, it allows characterization of the range of the perturbed operator $(P-\mu)$, i.e., the space $\mathrm{Im}(P-\mu)=\{\chi\in L^2(M,V)\mid \chi=(P-\mu)\psi \text{ for } \psi\in\mathcal{D}_-(P)\}$ as the L^2-complement of the eigenspace $\mathcal{N}_\mu(P)$. This allows to prove the compactness of the resolvent and the existence of a spectral gap:

Theorem 7.
(i) *The Dirac operator P with domain $\mathcal{D}_-(P)$ is a Fredholm operator.*
(ii) *For each $\mu\in\mathbb{R}$ the operator P induces an orthogonal decomposition*

$$L^2(M,V)=\mathcal{N}_\mu(P)\oplus\mathrm{Im}(P-\mu). \tag{6.1}$$

In particular, the index $Ind(P-\mu)=0$.

Proof.
(i) Let $\phi_j\in\mathrm{Im}(P)$ be a L^2 Cauchy sequence. Then $\phi_j=P\psi_j$, and without loss of generality we can choose $\psi_j\in(\mathcal{N}_0^\perp(P)\cap\mathcal{D}_-(P))$. By (5.13)

$$\|\psi_j-\psi_k\|^2_{H^1}\le C_0\|\phi_j-\phi_k\|^2_{L^2}\longrightarrow 0. \tag{6.2}$$

Therefore $\psi_j\to\psi$ in $H^1(M,V)$ and hence $\psi\in\mathcal{D}_-(P)$. This proves that the range $\mathrm{Im}(P)$ of P is closed in $L^2(M,V)$. Since its kernel $\mathcal{N}_0(P)$ is finite dimensional, P is Fredholm.

(ii) Since P is self-adjoint and Fredholm, so is $(P-\mu)$ for each $\mu\in\mathbb{R}$. From the closed range theorem we then infer that

$$\mathrm{Im}(P-\mu)=\mathrm{Ker}(P-\mu)^\perp, \tag{6.3}$$

which proves the decomposition (6.1). By self-adjointness the index vanishes. □

Lemma 8. *Let $\widetilde{\mu}\in\mathbb{R}$ satisfy $0<\widetilde{\mu}^2<1/C_0$, with C_0 given by the estimate (5.13).*
(i) *The $\widetilde{\mu}$-eigenspace of P is trivial, i.e. $\mathcal{N}_{\widetilde{\mu}}(P)=0$.*
(ii) *The corresponding resolvent is a compact operator*

$$(P-\widetilde{\mu})^{-1}:L^2(M,V)\longrightarrow L^2(M,V). \tag{6.4}$$

Proof.
(i) Since P is self-adjoint, $\mathcal{N}_{\widetilde{\mu}}(P)\subset(\mathcal{N}_0^\perp(P)\cap\mathcal{D}_-(P))$. By (5.13)

$$\|\psi\|^2_{L^2}\le C_0\|P\psi\|^2_{L^2}=C_0\,\widetilde{\mu}^2\|\psi\|^2_{L^2}\quad\forall\psi\in\mathcal{N}_{\widetilde{\mu}}(P), \tag{6.5}$$

and from $\widetilde{\mu}^2<1/C_0$ we infer that $\mathcal{N}_{\widetilde{\mu}}(P)=0$.

(ii) By (i), the operator $(P - \widetilde{\mu}) : \mathcal{D}_-(P) \to L^2(M, V)$ is a continuous bijection. The open mapping theorem guarantees the existence of a continuous resolvent $(P - \widetilde{\mu})^{-1}$: $L^2(M, V) \to H^1(M, V)$, satisfying

$$\|(P - \widetilde{\mu})^{-1}\phi\|^2_{H^1} \leq C_{\widetilde{\mu}} \|\phi\|^2_{L^2} \quad \forall \phi \in L^2(M, V). \tag{6.6}$$

If $\phi_j \in L^2(M, V)$ is a bounded sequence, then $\|(P - \widetilde{\mu})^{-1}\phi_j\|_{H^1}$ is bounded, too. Since the embedding $H^1(M, V) \hookrightarrow L^2(M, V)$ is compact, there exists a subsequence ϕ_{j_k} such that

$$(P - \widetilde{\mu})^{-1}\phi_{j_k} \longrightarrow \widehat{\psi} \text{ (strongly) in } L^2(M, V). \tag{6.7}$$

Hence the resolvent is a compact operator. □

Theorem 9. *The spectrum of the Dirac operator P with domain $\mathcal{D}_-(P)$ consists of entirely isolated real eigenvalues with finite multiplicity. It admits a discrete spectral resolution, i.e.*

$$L^2(M, V) = \bigoplus_{\lambda \in \operatorname{spec}(P)} \mathcal{N}_\lambda(P). \tag{6.8}$$

Proof. Since P is an operator with compact resolvent (cf. [7]), it has a purely discrete spectrum. By the Hilbert–Schmidt theorem there exists a spectral resolution $(\varphi_k)_{k \in \mathbb{N}}$ such that

$$(P - \mu)^{-1}\varphi_k = \nu_k \varphi_k \text{ with } \varphi_k \in \mathcal{D}_-(P) \text{ and } |\nu_k| \longrightarrow 0. \tag{6.9}$$

Then, by construction,

$$P\varphi_k = \lambda_k \varphi_k. \tag{6.10}$$

where $\lambda_k = (1/\nu_k + \mu)$. This implies the desired spectral resolution (6.8) of P. □

In addition to this structural result on the spectrum of the Dirac operator under the respective boundary conditions $\psi|_{\partial M} \in \Gamma^{\text{loc}}_-$ and $\psi|_{\partial M} \in \Gamma^{\text{APS}}_-$, our approach allows one to give estimates for first eigenvalue of P, i.e. the eigenvalue of least absolute value. If ψ is a eigenspinor we infer from (3.9) and (4.5) that

$$\|\nabla\psi\|^2_{L^2} \leq \lambda^2 \|\psi\|^2_{L^2} - \langle\langle \psi, \mathcal{R}\psi \rangle\rangle - \int_{\partial M} \left(\frac{1}{2}\mathcal{S}_\partial - \epsilon\right) \langle \psi, \psi \rangle \, d\mu_\partial \quad \forall \psi \in \mathcal{N}_\lambda(P). \tag{6.11}$$

As far as the boundary condition $\psi|_{\partial M} \in \Gamma^{\text{loc}}_-$ is concerned $\epsilon = 0$; for the case $\psi|_{\partial M} \in \Gamma^{\text{APS}}_-$ we can choose $\epsilon > 0$ arbitrarily small. The curvature endomorphism is a symmetric operator in $\mathcal{R} \in \operatorname{End}(V)$ which has for each $p \in M$ a smallest eigenvalue $\rho_0(p)$.

Theorem 10. *Let $(V, \langle\ ,\ \rangle, \gamma, \nabla)$ be a Dirac bundle over a compact manifold M with boundary and assume that the trace of the mean curvature is strictly positive, i.e. $\mathcal{S}_\partial(p) >$*

0 for all $p \in \partial M$. Let spec(P) *be the spectrum of the Dirac operator with domain $\mathcal{D}_-(P)$. Writing this as an ordered set*

$$\operatorname{spec}(P) = \{\lambda_i \mid i \in \mathbb{N} \text{ with } 0 \leq |\lambda_1| \leq |\lambda_2| \leq \cdots\},$$

the curvature endomorphism gives a lower bound for the first eigenvalue:

$$(\lambda_1)^2 \geq \min_{p \in M} \rho_0(p). \tag{6.12}$$

If $\mathcal{D}_-(P)$ is understood with respect to the local boundary condition $\psi|_{\partial M} \in \Gamma_-^{\text{loc}}$ this also holds if $\mathcal{S}_\partial$ is non-negative on ∂M.

Proof. Under the assumption that $\mathcal{S}_\partial$ is strictly positive on ∂M the value of ϵ can be chosen sufficiently small so that $(\mathcal{S}_\partial(p) - 2\epsilon) \geq 0$ for all $p \in \partial M$. Then (6.11) implies that

$$0 \leq \left(\lambda^2 - \int_M \langle \psi, \mathcal{R}\psi \rangle \,\mathrm{d}\mu\right) \leq \left(\lambda^2 - \int_M \rho_0(p) \langle \psi, \psi \rangle \,\mathrm{d}\mu\right), \tag{6.13}$$

holding for all $\psi \in \mathcal{N}_\lambda(P)$ with $\|\psi\|_{L^2} = 1$. Since $\rho_0(p)$ is continuous and M is compact, this implies the estimate (6.12). As far as the local boundary condition $\psi|_{\partial M} \in \Gamma_-^{\text{loc}}$ is concerned we have $\epsilon = 0$. This implies that (6.13) also holds under the assumption $\mathcal{S}(p) \geq 0$ for all $p \in \partial M$. □

Having characterized the spectrum of P it is obvious to formulate the corresponding spectral theorem for the Dirac–Laplace operator P^2. Therefore we set

$$\mathcal{D}_-(P^2) := \{\psi \in L^2(M, V) \mid \psi \in \mathcal{D}_-(P) \text{ and } P\psi \in \mathcal{D}_-(P)\}. \tag{6.14}$$

Corollary 11. *The Dirac–Laplace operator P^2 with domain $\mathcal{D}_-(P^2)$ is a self-adjoint positive operator. It admits a discrete spectral resolution, i.e.*

$$L^2(M, V) = \bigoplus_{k \in \mathbb{N}} \mathcal{N}_{\lambda_k}(P^2) \tag{6.15}$$

where $\mathcal{N}_\mu(P^2) := \{\phi \in \mathcal{D}_-(P^2) \mid P\psi \in \mathcal{D}_-(P)$ and $(P^2 - \mu)\phi = 0\}$. Moreover, P^2 is the generator of a contracting semigroup $\exp(-P^2)$ *in $L^2(M, V)$.*

Proof. It is immediate from Theorem 4 that the Dirac–Laplace operator with domain $\mathcal{D}_-(P^2)$ is self-adjoint. Moreover,

$$\langle\langle \psi, P^2\psi \rangle\rangle = \|P\psi\|^2_{L^2} \quad \forall \psi \in \mathcal{D}_-(P^2), \tag{6.16}$$

which implies that P^2 is a positive operator. By Theorem 9 it has a discrete spectral resolution with

$$\operatorname{spec}(P^2) = \{\lambda^2 \in \mathbb{R} \mid \lambda \in \operatorname{spec}(P)\}. \tag{6.17}$$

Therefore the operator $(-P^2)$ is dissipative and densely defined in $L^2(M, V)$, and the existence of a semi-group $\exp(-P^2)$ follows from the Theorem of Lumer-Phillips [9]. □

7. Regularity results

The H^k Sobolev norm on $C^\infty(M, V)$ can be defined inductively by

$$\|\psi\|^2_{H^k} = \sum_{|\beta|=k} \|\nabla_\beta \psi\|^2_{L^2} + \|\psi\|^2_{H^{k-1}}. \tag{7.1}$$

Here β is a multi-index, the derivatives ∇_β are taken with respect to a local orthonormal frame $\{e_1, \dots, e_n\}$, and the norm $\|\nabla_\beta \psi\|_{L^2}$ is understood as the L^2 integral over M with an appropriate partition of unity. Denoting the corresponding completion by $H^k(M, V)$, the Sobolev embedding theorem states that $H^k(M, V) \hookrightarrow C^s(M, V)$ for $k > s + \frac{1}{2}\dim M$.

Lemma 12. *If $\psi \in \mathcal{N}_\lambda(P)$ satisfies either the local boundary condition $\psi|_{\partial M} \in \Gamma^{\text{loc}}_-$ or the global one $\psi|_{\partial M} \in \Gamma^{\text{APS}}_-$, then ψ is smooth. In particular $\psi \in H^k(M, V)$ for all $k \in \mathbb{N}$.*

Proof. First we observe that P commutes with the covariant derivatives ∇_{e_j} up to lower order terms, i.e.

$$P\nabla_{e_j}\psi - \nabla_{e_j}P\psi = \sum_{i=1,\dots,n} (\gamma(\nabla_{e_j}e_i)\nabla_{e_j} + \gamma(e_i)\nabla_{\nabla_{e_i}e_j} + \gamma(e_i)\mathcal{R}(e_i, e_j))\psi \tag{7.2}$$

for $j = 1, \dots, n$. Since the geometry of M is bounded the Christoffel symbols appearing in (7.2) are bounded too, and we can estimate

$$\sum_{j=1,\dots,n} \|P\nabla_{e_j}\psi\|^2_{L^2} \le \sum_{j=1,\dots,n} \|\nabla_{e_j}P\psi\|^2_{L^2} + C\|\psi\|^2_{H^1}. \tag{7.3}$$

On the basis of this we can apply the argument of Eq. (2.11) to $\sum_{j=1,\dots,n} \nabla_{e_j}\psi$ which yields

$$\sum_{|\beta|=2} \|\nabla_\beta\psi\|^2_{L^2} \le \sum_{j=1,\dots,n} \|\nabla_{e_j}P\psi\|^2_{L^2} + C'\|\psi\|^2_{H^1} + \sum_{j=1,\dots,n} \int_{\partial M} \langle \nabla_{e_j}\psi, A\nabla_{e_j}\psi\rangle \, \mathrm{d}\mu_\partial. \tag{7.4}$$

To control the boundary integral under the local boundary condition Γ^{loc}_- let $\widetilde{N}$ be a smooth extension of the normal field. From $\widetilde{\pi}_\pm = \frac{1}{2}(F\gamma(\widetilde{N}) \pm \mathrm{Id})$ we infer that

$$\nabla_{e_j}\widetilde{\pi}_-(\psi) = \widetilde{\pi}_-(\nabla_{e_j}\psi) - \tfrac{1}{2}F\gamma(\nabla_{e_j}\widetilde{N})\psi \quad \forall j = 1, \dots, n. \tag{7.5}$$

With (3.6) – here understood accordingly for the extension $\widetilde{\pi}_\pm$ – we then get

$$\begin{aligned} &\langle \nabla_{e_j}\widetilde{\pi}_-(\psi), A\nabla_{e_j}\widetilde{\pi}_-(\psi)\rangle \\ &\quad = \langle \widetilde{\pi}_-(\nabla_{e_j}\psi), \widetilde{\pi}_+(A\nabla_{e_j}\psi)\rangle + \tfrac{1}{2}(\langle \widetilde{\pi}_-(\nabla_{e_j}\psi), \widetilde{S}_\partial\widetilde{\Gamma}\,\nabla_{e_j}\psi\rangle \\ &\qquad - \langle \widetilde{\pi}_-(\nabla_{e_j}\psi), AF\gamma(\nabla_{e_j}\widetilde{N})\psi\rangle - \langle F\gamma(\nabla_{e_j}\widetilde{N})\,\psi, A\widetilde{\pi}_-(\nabla_{e_j}\psi)\rangle). \end{aligned} \tag{7.6}$$

Under the restriction to the boundary ∂M the projections $\tilde{\pi}_+$ and $\tilde{\pi}_-$ become orthogonal, and the operators $\tilde{S}_\partial \tilde{\Gamma}$ and $F\gamma(\nabla_{e_j}\tilde{N})$ have a bounded trace. Thus we have

$$\sum_{j=1,\dots,n} \int_{\partial M} \langle \nabla_{e_j}\psi, A\nabla_{e_j}\psi\rangle \, \mathrm{d}\mu_\partial \le C \left(\sum_{j=1,\dots,n} \|\nabla_{e_j}\psi\|^2_{L^2(\partial M)} + \|\psi\|^2_{L^2(\partial M)} \right), \tag{7.7}$$

holding for all ψ which satisfy the boundary condition $\psi|_{\partial M} \in \Gamma_-^{\mathrm{loc}}$. Since the restriction to the boundary is an compact linear map from $H^1(M, V)$ to $L^2(\partial M, V_\partial)$, the Ehrling inequality (3.10) implies that

$$\left(\sum_{j=1,\dots,n} \|\nabla_{e_j}\psi\|^2_{L^2(\partial M)} + \|\psi\|^2_{L^2(\partial M)} \right) \le \tfrac{1}{2}\|\psi\|^2_{H^2(M)} + C\|\psi\|^2_{L^2(M)} \quad \forall \psi \in C^\infty(M, V). \tag{7.8}$$

For ψ subject to the Atiyah–Patodi–Singer boundary condition $\psi|_{\partial M} \in \Gamma_-^{\mathrm{APS}}$ the same type of arguments applies. Therefore we have

$$\sum_{|\beta|=2} \|\nabla_\beta \psi\|^2_{L^2} \le \frac{1}{2}\|\psi\|^2_{H^2(M)} + C_2(\|P\psi\|^2_{H^1} + \|\psi\|^2_{H^1}) + \epsilon\|\psi\|^2_{H^2(M)}, \tag{7.9}$$

holding for each $\epsilon > 0$ under either of the boundary conditions $\psi \in \Gamma_-$. By iteration of these arguments we can derive for an arbitrary $k \in \mathbb{N}$ a corresponding estimate, reading

$$\sum_{|\beta|=k} \|\nabla_\beta \psi\|^2_{L^2} \le \frac{1}{2}\|\psi\|^2_{H^k(M)} + C_k(\|P\psi\|^2_{H^{k-1}} + \|\psi\|^2_{H^{k-1}}). \tag{7.10}$$

If, in particular, $P\psi = \lambda\psi$ this implies that

$$\|\psi\|^2_{H^k} \le 2(\lambda^2 + C_k)\|\psi\|^2_{H^{k-1}} \quad \forall \psi \in \mathcal{N}_\lambda(P), \tag{7.11}$$

and by iteration we end up with

$$\|\psi\|^2_{H^k} \le \tilde{C}(\lambda)\|\psi\|^2_{L^2} \quad \forall \psi \in \mathcal{N}_\lambda(P). \tag{7.12}$$

Consequently, $\psi \in \mathcal{N}_\lambda(P)$ implies that $\psi \in H^k(M, V)$ for all $k \in \mathbb{N}$. From the Sobolev embedding theorem we then infer that ψ is smooth. □

Corollary 13. *The splitting* (6.1) *respects the* C^∞ *structure, i.e.,*

$$C^\infty(M, V) = \mathcal{N}_\lambda(P) \oplus \mathrm{Im}((P - \lambda)|_{C^\infty(M,V)}) \,. \tag{7.13}$$

Proof. Intersecting the decomposition (6.1) with $C^\infty(M, V)$ and using the preceding lemma implies that each $\phi \in C^\infty(M, V)$ uniquely splits into $\phi = \phi_0 + (P - \lambda)\chi$ with $\phi_0 \in \mathcal{N}_\lambda(P) \subset C^\infty(M, V)$ and $(P - \lambda)\chi \in C^\infty(M, V)$. Using (7.10) and (7.8) we get $\chi \in H^k(M, V)$ for all $k \in \mathbb{N}$, which implies that χ is smooth. □

Elliptic regularity, could be established as well by checking the Lopatinskii–Shapiro condition [8] for the boundary value problem in view. For the general case of pseudo-differential boundary problems, cf. [3].

8. Applications: Boundary value problems and Hodge type decomposition

To this end we consider a general inhomogeneous boundary value problem for the Dirac operator, reading

$$\begin{aligned} P\phi &= \psi && \text{on } M \\ \pi_+(\phi) &= \pi_+(\rho) && \text{on } \partial M. \end{aligned} \tag{8.1}$$

Lemma 14. *For each $\psi \in L^2(M, V)$ and $\rho \in H^1(M, V)$ satisfying the integrability condition*

$$\langle\langle \psi, \chi \rangle\rangle + \int_{\partial M} \langle \gamma(N)\rho, \chi \rangle \, d\mu_\partial = 0 \quad \forall \chi \in \mathcal{N}_0(P). \tag{8.2}$$

the boundary problem (8.1) *has a solution, which is unique up to an arbitrary $\widetilde{\chi} \in \mathcal{N}_0(P)$.*

Proof. Given $\psi \in L^2(M, V)$ and $\rho \in H^1(M, V)$ we consider the field $(\psi - P\rho) \in L^2(M, V)$. By Theorem 7, the index of the boundary value problem (8.1) vanishes. Therefore $(\psi - P\rho)$ is in the range $\mathrm{Im}(P)$ of the Dirac operator if and only if $(\psi - P\rho) \in \mathcal{N}_0(P)^\perp$. That is

$$\langle\langle (\psi - P\rho), \chi \rangle\rangle = 0 \quad \forall \chi \in \mathcal{N}_0(P), \tag{8.3}$$

which is, by (2.5), equivalent to the integrability condition (8.2). Therefore

$$(\psi - P\rho) = P\widetilde{\phi} \ \text{ with } \ \widetilde{\phi} \in \mathcal{D}_-(P). \tag{8.4}$$

Choosing $\phi := \widetilde{\phi} + \rho$ we have $\phi \in H^1(M, V)$. By construction it satisfies $P\phi = \psi$ and $\pi_+(\phi) = \pi_+(\rho)$, and hence it is a solution of the boundary value problem. For any other solution of the form $\phi + \widetilde{\chi}$ we infer from (8.1) that $P\widetilde{\chi} = 0$ and $\pi_+(\widetilde{\chi}) = 0$. □

Finally we make an attempt to construct a Hodge type decomposition for the Dirac bundle $(V, <, >, \gamma, \nabla)$. To do so we define the operator $d_P = \frac{1}{2}(\mathrm{Id} - F)P$ on $C^\infty(M, V)$. Since F anti-commutes with P it follows that $(d_P)^2 = 0$. Therefore we have an elliptic complex

$$\ldots \xrightarrow{d_P} C_0^\infty(M, V) \xrightarrow{d_P} C_0^\infty(M, V) \xrightarrow{d_P} \ldots \tag{8.5}$$

Using (2.5), it follows that $d_P^* := \frac{1}{2}(\mathrm{Id} + F)P$ is the formal adjoint, satisfying

$$\langle\langle d_P\psi, \phi \rangle\rangle = \langle\langle \psi, d_P^*\phi \rangle\rangle \quad \forall \phi, \psi \in C_0^\infty(M, V). \tag{8.6}$$

However, the boundary conditions $\psi \in \Gamma_\pm$ do not serve as an absolute or relative boundary condition for this complex – neither in the case $\Gamma_\pm^{\mathrm{loc}}$ or $\Gamma_\pm^{\mathrm{APS}}$. Nevertheless it is possible to construct a Hodge-type decomposition for bundle $L^2(M, V)$ in terms of this complex.

Lemma 15. *The space $L^2(M, V)$ allows for an orthogonal decomposition*

$$L^2(M, V) = d_P(H_-^1(M, V)) \oplus d_P^*(H_-^1(M, V)) \oplus \mathcal{N}_0(P)\,, \tag{8.7}$$

where $H^1_-(M, V) = \{\phi \in H^1(M, V) \mid \pi_+(\phi) = 0\}$. *Each* $\psi \in L^2(M, V)$ *uniquely splits into*

$$\psi = d_P\phi + d_P^*\phi + \chi \quad \text{with} \quad \phi \in \mathcal{D}_-(P), \quad \chi \in \mathcal{N}_0(P). \tag{8.8}$$

Proof. By (6.1), ψ splits into $\psi = P\phi + \chi$ with $\phi \in \mathcal{D}_-(P)$ and $\chi \in \mathcal{N}_0(P)$. Then $P = d_P + d_P^*$ which implies the decomposition (8.8). □

As far as the boundary conditions are concerned, this decomposition yields $\phi|_{\partial M} \in \Gamma_-$. However, there is no control about the boundary behavior of the other components $d_P\phi$ and $d_P^*\phi$, but only the sum $((d_P + d_P^*)\phi)|_{\partial M}$ will be in Γ_-.

Acknowledgements

We are grateful to an anonymous referee for pointing out to us a number of classical results on the field. One of the authors (GS) likes to thank the Forschungsinstitut für Mathematik, ETH Zürich, for its hospitality.

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